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# Classical lattice $W$ algebras and integrable systems

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**Abstract.** Several aspects of the lattice $W_N$ algebra are studied. Motivated by the fact that the Lotka–Volterra model can be written in terms of a current of the lattice Virasoro algebra (the Faddeev–Takhtajan–Volkov algebra), integrable dynamical models on the lattice have been formulated as a model associated with the lattice $W_3$ algebra.

## 1. Introduction

The classical $W_N$ algebra can be constructed from the free fields $r_i(x)$ for $i = 1, 2, \ldots, N$ satisfying the Poisson algebra [1–3],

$$\{r_i(x), r_j(y)\} = \left(-\delta_{i,j} + \frac{1}{N}\right)\delta'(x-y). \tag{1.1}$$

It is well known that the Poisson structure (1.1) is related to the $N$-reduced KP hierarchy, and that the Poisson map from the free fields to the pseudodifferential operator $L$ is defined as

$$L = \partial^N + \sum_{i=1}^{N-1} w_i(x)\partial^{N-1-i} = (\partial + r_1(x))(\partial + r_2(x))\ldots(\partial + r_N(x)) \tag{1.2}$$

with $\partial = \partial/\partial x$, and

$$\sum_{i=1}^{N} r_i(x) = 0. \tag{1.3}$$

The fields $w_i(x)$ for $i = 1, 2, \ldots, N-1$ constitute the classical $W_N$ algebra, and the relation between $w_i(x)$ and $r_j(x)$ is called the Miura transformation. Among the Poisson relations for the fields $w_i(x)$, one sees that the field $w_1(x)$ constitute the Virasoro algebra,

$$\{w_1(x), w_1(y)\} = \left(w_1(x)\partial + \partial w_1(x) + \frac{N(N^2-1)}{12}\partial^3\right)\delta(x-y). \tag{1.4}$$

This fact shows that the $W_N$ algebra includes the Virasoro algebra as a subalgebra.

Recently the difference analogue of the $W_N$ algebra in both the classical and quantum cases has received much attention [4–6]. The $q$-deformation of the free field realization (1.1)

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was given in [7] as algebra for the $q$-deformed free fields $\Lambda_i(z)$ for $i = 1, 2, \ldots, N$;

$$\{\Lambda_i(z), \Lambda_i(w)\} = \sum_{m=-\infty}^{\infty} \left(\frac{w}{z}\right)^m \frac{(1-q^m)(1-q^{m(N-1)})}{1-q^{mN}} \Lambda_i(z)\Lambda_i(w) \tag{1.5a}$$

$$\{\Lambda_i(z), \Lambda_j(w)\} = -\sum_{m=-\infty}^{\infty} \left(\frac{wq^{N+i-j-1}}{z}\right)^m \frac{(1-q^m)^2}{1-q^{mN}} \Lambda_i(z)\Lambda_j(w) \qquad \text{for } i < j. \tag{1.5b}$$

In terms of these deformed free fields, the difference analogue of the pseudodifferential operator (1.2) is defined as

$$\begin{aligned} L &= (D - \Lambda_1(z))(D - \Lambda_2(z)) \ldots (D - \Lambda_N(z)) \\ &= D^N - t_1(z)D^{N-1} + \cdots + (-)^{N-1} t_{N-1}(z) D + (-)^N. \end{aligned} \tag{1.6}$$

Here $D$ denotes a $q$-difference operator,

$$(Df)(z) = f(zq)$$

and the $q$-deformed free fields $\Lambda_i(z)$ satisfy a condition,

$$\Lambda_1(z)\Lambda_2(z) \ldots \Lambda_N(z) = 1. \tag{1.7}$$

Identities among $t_i(z)$ and $\Lambda_i(z)$ are called the $q$-deformation of the Miura transformation (1.2). With the difference operator $L$ (1.6), we can define the difference analogue of the Korteweg–de Vries (KdV)-type hierarchy [7].

On the other hand, there is another deformation of the $W_N$ algebra, i.e. the lattice version of the $W_N$ algebra, which may be related to the $\mathfrak{sl}_N$ Toda theory on the lattice [8–14]. The most famous example is the Lotka–Volterra model. It is known that the Lotka–Volterra model reduces to the KdV equation in the continuum limit, and that it can be formulated in terms of the lattice Virasoro algebra (the Faddeev–Takhtajan–Volkov algebra) [8–11]. In this paper, following a method of [15], we study the lattice $W_N$ algebra. Introducing the Lax matrix, we also define the integrable models associated with the lattice $W_N$ algebra.

This paper is organized as follows. In section 2 the lattice Virasoro algebra is formulated following [15]. The relationship with Frenkel's deformed KdV equation becomes clear. The integrability of the Lotka–Volterra model is reformulated with the lattice Virasoro algebra. In section 3 we generalized a method of [15] to the lattice $W_3$ algebra. Although the definition of the free field on the lattice is different from the previously known method [12–14], the resulting $W_3$ algebra is exactly the same. It is also discussed that the integrable models are associated with the lattice $W_3$ algebra. By introducing the $3 \times 3$ 'local' Lax matrix, we define the integrable systems on the lattice. It is shown that, in the continuum limit, the dynamical models on the lattice reduce to the nonlinear differential equations, which belong to the Boussinesq hierarchy. The last section is devoted to the concluding remarks.

## 2. Lattice KdV hierarchy

### 2.1. Lattice Virasoro algebra

We consider the deformation of the KdV hierarchy ($N = 2$ in (1.6)), in which the $q$-difference operator $L$ (1.6) is written as

$$L = D^2 - t(z)D + 1 = (D - \Lambda(z))\left(D - \frac{1}{\Lambda(z)}\right). \tag{2.1}$$

The second equality indicates that the field $t(z)$ is related to the $q$-deformed free field $\Lambda(z)$ by the $q$-deformed Miura transformation,

$$t(z) = \Lambda(z) + \frac{1}{\Lambda(qz)}. \tag{2.2}$$

As the Poisson relation (1.5) for $N = 2$ case is given by

$$\{\Lambda(z), \Lambda(w)\} = \sum_{m=-\infty}^{\infty} \frac{1-q^m}{1+q^m} \left(\frac{w}{z}\right)^m \Lambda(z)\Lambda(w) \tag{2.3}$$

we see that the field $t(z)$ satisfies a relation,

$$\{t(z), t(w)\} = \sum_{m=-\infty}^{\infty} \frac{1-q^m}{1+q^m} \left(\frac{w}{z}\right)^m t(z)t(w) + \delta\left(\frac{qw}{z}\right) - \delta\left(\frac{w}{qz}\right). \tag{2.4}$$

Hereafter, in the $q$-difference case, the delta function $\delta(z)$ is denoted as

$$\delta(z) = \sum_{m=-\infty}^{\infty} z^m. \tag{2.5}$$

We note that the field $\ell(z)$, defined by

$$\ell(z) = \frac{1}{t(z)t(qz)} = \frac{\Lambda(qz)\Lambda(q^2z)}{(\Lambda(z)\Lambda(qz)+1)(\Lambda(qz)\Lambda(q^2z)+1)} \tag{2.6}$$

satisfies a simple Poisson relation;

$$\begin{aligned}\{\ell(z), \ell(w)\} = {} & \delta\left(\frac{q^2w}{z}\right)\ell(w)\ell(qw)\ell(q^2w) + \delta\left(\frac{qw}{z}\right)\ell(z)\ell(w)(\ell(z)+\ell(w)-1) \\ & -\delta\left(\frac{w}{qz}\right)\ell(z)\ell(w)(\ell(z)+\ell(w)-1) - \delta\left(\frac{w}{q^2z}\right)\ell(z)\ell(qz)\ell(q^2z).\end{aligned} \tag{2.7}$$

One sees that the Poisson algebra for the field $\ell(z)$ does not contain the infinite sum terms.

Following a strategy of [15], we consider the dynamical variables $v_k$ on the lattice. The Poisson algebra for $v_k$ is supposed to be

$$\{v_k, v_l\} = \begin{cases} \eta(-)^{k-l} v_k v_l & \text{for } k < l \\ 0 & \text{for } k = l \\ -\{v_l, v_k\} & \text{for } k > l \end{cases} \tag{2.8}$$

where $\eta$ is arbitrary. The dynamical variable $v_k$ naively corresponds to the $q$-deformed free field $\Lambda(zq^k)$; (2.3) reduces to (2.8) with a proper choice of a parameter $q$ [15].

We shall define the Miura transformation on the lattice in the same way with (2.2) as

$$s_k = v_k + \frac{1}{v_{k+1}}. \tag{2.9}$$

The Poisson relations for variables $s_k$ are computed from (2.8) and (2.9) as,

$$\{s_k, s_l\} = \eta(-)^{k-l} s_k s_l + \eta\delta_{k+1,l} \qquad \text{for } k < l. \tag{2.10}$$

The above Poisson algebra is a lattice analogue of the algebra (2.4).

To relate with the known Poisson algebra, we introduce a new variable $S_k$ by

$$S_k = \frac{1}{s_k s_{k+1}}. \tag{2.11}$$

As the Poisson algebra (2.7) for the field $\ell(z)$ does not include the infinite-sum terms, the non-trivial Poisson relations among dynamical variables $S_k$ are simply given as

$$\{S_k, S_{k+2}\} = \eta S_k S_{k+1} S_{k+2} \tag{2.12a}$$

$$\{S_k, S_{k+1}\} = -\eta S_k S_{k+1}(1 - S_k - S_{k+1}). \tag{2.12b}$$

The algebra (2.12) is a lattice analogue of (2.7), and is called the lattice Virasoro algebra, or the Faddeev–Takhtajan–Volkov (FTV) algebra [8–11]. We remark that the FTV algebra originally appeared in studies of the lattice Liouville theory and the massless sine-Gordon model, which is related to the lattice KdV equation [16].

We note that transformation (2.11) is rewritten using the lattice Miura transformation (2.9) as

$$S_k = \frac{\alpha_{k+1}}{(1+\alpha_k)(1+\alpha_{k+1})}. \tag{2.13}$$

Here variables $\alpha_k$ are dynamical variables defined by

$$\alpha_k = v_k v_{k+1} \tag{2.14}$$

and using the Poisson algebra (2.8), they are proved to satisfy the local Poisson relation,

$$\{\alpha_k, \alpha_{k+1}\} = -\eta \alpha_k \alpha_{k+1}. \tag{2.15}$$

Transformation (2.13) was introduced in [9] as a lattice analogue of the Miura transformation.

### 2.2. Hamiltonian structure

We review the results of [8–11] for the classical integrable models associated with the FTV algebra (2.12). We define the Lax matrix $\tilde{\mathbf{L}}_n(\lambda)$ for $n \in \mathbb{Z}$ as a $2 \times 2$ matrix,

$$\tilde{\mathbf{L}}_n(\lambda) = \frac{1}{\sqrt{S_n}} \begin{pmatrix} \lambda & -S_n \\ 1 & 0 \end{pmatrix}. \tag{2.16}$$

Here $\lambda$ is called the spectral parameter. We also introduce the transfer matrix $t(\lambda)$ as the product of the Lax matrices;

$$\begin{aligned} t(\lambda) &= \operatorname{Tr} \mathbf{T}(\lambda) \\ &= \operatorname{Tr}\left( \overset{\curvearrowleft}{\prod_k} \tilde{\mathbf{L}}_k(\lambda) \right) = \operatorname{Tr}(\ldots \tilde{\mathbf{L}}_{n+1}(\lambda) \tilde{\mathbf{L}}_n(\lambda) \tilde{\mathbf{L}}_{n-1}(\lambda) \ldots) \end{aligned} \tag{2.17}$$

where a matrix $\mathbf{T}(\lambda)$ is the monodromy matrix. We suppose that a lattice is infinite or periodic. To study the integrable model associated with the transfer matrix $t(\lambda)$ (2.17), we gauge-transform the Lax matrix (2.16) as

$$\mathbf{L}_{n+1}(\lambda) = \mathbf{\Omega}_{n+1}(\lambda) \tilde{\mathbf{L}}_n(\lambda) \mathbf{\Omega}_n(\lambda)^{-1} \tag{2.18}$$

with $\mathbf{\Omega}_n(\lambda)$ defined by

$$\mathbf{\Omega}_n(\lambda) = \begin{pmatrix} (\lambda^2-1)^{-1/4}(1+\alpha_n)^{1/2} & 0 \\ 0 & (\lambda^2-1)^{1/4}(1+\alpha_n)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & -\lambda(\alpha_n+1)^{-1} \\ 0 & 1 \end{pmatrix}. \tag{2.19}$$

As a result, we obtain the local $L$-matrix,

$$\mathbf{L}_n(\lambda) = \begin{pmatrix} \lambda \alpha_n^{1/2} & (\lambda^2-1)^{1/2} \alpha_n^{1/2} \\ (\lambda^2-1)^{1/2} \alpha_n^{-1/2} & \lambda \alpha_n^{-1/2} \end{pmatrix}. \tag{2.20}$$

Owing to the gauge transformation (2.18), the Poisson algebra for the Lax matrix becomes quite simple, and the non-trivial Poisson relation is written as

$$\{\mathbf{L}_n(\lambda) \overset{\otimes}{,} \mathbf{L}_{n+1}(\mu)\} = -\frac{\eta}{4}\boldsymbol{\sigma}_3\mathbf{L}_n(\lambda) \otimes \boldsymbol{\sigma}_3\mathbf{L}_{n+1}(\mu) \tag{2.21}$$

where $\boldsymbol{\sigma}_3$ is the Pauli matrix,

$$\boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To see the integrability of the classical model, we need another identity for the Lax matrix,

$$-\boldsymbol{\sigma}_3\mathbf{L}_n(\lambda)\otimes\mathbf{L}_n(\mu)\boldsymbol{\sigma}_3+\mathbf{L}_n(\lambda)\boldsymbol{\sigma}_3\otimes\boldsymbol{\sigma}_3\mathbf{L}_n(\mu) = 2[\mathbf{r}(\lambda, \mu), \mathbf{L}_n(\lambda)\otimes\mathbf{L}_n(\mu)] \tag{2.22}$$

where the $r$-matrix is defined by

$$\mathbf{r}(\lambda, \mu) = \begin{pmatrix} a & & & \\ & 0 & b & \\ & b & 0 & \\ & & & a \end{pmatrix}. \tag{2.23}$$

Each element is given as

$$a = \frac{\theta + \theta^{-1}}{\theta - \theta^{-1}} \qquad b = \frac{2}{\theta - \theta^{-1}}$$

with a modified spectral parameter $\theta$,

$$\theta = \sqrt{\frac{1 - \mu^{-2}}{1 - \lambda^{-2}}}.$$

We note that the $r$-matrix (2.23) is a solution of the classical Yang–Baxter equation, and a quasiclassical limit of the $R$-matrix for the six vertex model. From identities (2.21) and (2.22), we see that the monodromy matrix $\mathbf{T}(\lambda)$ satisfies a usual Poisson relation,

$$\{\mathbf{T}(\lambda) \overset{\otimes}{,} \mathbf{T}(\mu)\} = -\frac{\eta}{2}[\mathbf{r}(\lambda, \mu), \mathbf{T}(\lambda) \otimes \mathbf{T}(\mu)] \tag{2.24}$$

which proves the Poisson commutativity of the transfer matrix $t(\lambda)$ (2.17),

$$\{t(\lambda), t(\mu)\} = 0. \tag{2.25}$$

In conclusion, we have obtained the transfer matrix which generates the integrable Hamiltonians associated with the FTV algebra.

Once we have obtained the Poisson commutative transfer matrix, a set of the integrable Hamiltonians is given by expanding the transfer matrix $t(\lambda)$ (2.17) by a spectral parameter $\lambda$;

$$t(\lambda) = \exp(-\mathcal{H}_0)(\lambda^M + \lambda^{M-2}\mathcal{H}_1 + \lambda^{M-4}\mathcal{H}'_2 + \lambda^{M-6}\mathcal{H}'_3 + \cdots) \tag{2.26}$$

where $M$ denotes the size of the system. Noted are the explicit forms for some conserved quantities;

$$\mathcal{H}_0 = \tfrac{1}{2}\sum_n \log S_n \tag{2.27a}$$

$$\mathcal{H}_1 = -\sum_n S_n \tag{2.27b}$$

$$\begin{aligned}\mathcal{H}_2 &= \tfrac{1}{2}(\mathcal{H}_1)^2 - \mathcal{H}'_2 \\ &= \sum_n (\tfrac{1}{2}S_n^2 + S_n S_{n+1})\end{aligned} \tag{2.27c}$$

$$\begin{aligned}\mathcal{H}_3 &= \mathcal{H}_1\mathcal{H}_2' - \tfrac{1}{3}(\mathcal{H}_1)^3 - \mathcal{H}_3' \\ &= \sum_n \left(\tfrac{1}{3}S_n^3 + S_nS_{n+1}(S_n + S_{n+1} + S_{n+2})\right).\end{aligned} \tag{2.27d}$$

We consider the time evolutions associated with the Hamiltonian $\mathcal{H}_m$ by

$$\frac{\mathrm{d}S_n}{\mathrm{d}t_m} = \{\mathcal{H}_m, S_n\}. \tag{2.28}$$

One sees that the equation of motion for $\mathcal{H}_0$ (2.27*a*) is then given by

$$\frac{\mathrm{d}S_n}{\mathrm{d}t_0} = -\eta S_n(S_{n+1} - S_{n-1}) \tag{2.29}$$

which is called the Lotka–Volterra (LV) model. Also we obtain the equation of motion for $\mathcal{H}_1$ (2.27*b*) as

$$\frac{\mathrm{d}S_n}{\mathrm{d}t_1} = \eta(S_nS_{n+1}(S_n + S_{n+1} + S_{n+2} - 1) - S_nS_{n-1}(S_n + S_{n-1} + S_{n-2} - 1)) \tag{2.30}$$

which is a higher-order equation of motion for the Volterra hierarchy.

### 2.3. Continuum limit

We shall consider the continuum limit of the integrable dynamical equations on the lattice, (2.29) and (2.30). To this end, we set the free variable $v_k$ (2.8) on the lattice as

$$v_{n+i} \to \exp(\epsilon r(x - \mathrm{i}\epsilon)) \qquad \text{for } i = 0, \pm 1, \pm 2, \ldots \tag{2.31}$$

where $r(x)$ denotes a free field (1.1), and $\epsilon$ is an infinitesimal parameter. In this limit we see that, from (2.13) and (2.14), the Volterra's dynamical variable $S_n$ reduces to

$$S_n \to \frac{1}{4} + \frac{\epsilon^2}{4}w(x) - \frac{\epsilon^3}{4}w'(x) + \mathrm{O}(\epsilon^4) \tag{2.32}$$

where a field $w(x)$ is given from $r(x)$ by the Miura transformation,

$$w(x) = -r(x)^2 - r'(x) \tag{2.33}$$

and $'$ denotes a derivative with respect to $x$. To simplify our notation, we also rescale the time flows $t_0$ and $t_1$ (2.28) as

$$\frac{\mathrm{d}}{\mathrm{d}t_0} \to \eta\frac{\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}\tau_0} \tag{2.34a}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t_0} + \frac{\mathrm{d}}{\mathrm{d}t_1} \to -\eta\frac{\epsilon^3}{8}\frac{\mathrm{d}}{\mathrm{d}\tau_1}. \tag{2.34b}$$

We thus obtain the integrable nonlinear differential equations as continuum limits of (2.29) and (2.30), respectively,

$$\frac{\mathrm{d}w(x)}{\mathrm{d}\tau_0} = w'(x) \tag{2.35a}$$

$$\frac{\mathrm{d}w(x)}{\mathrm{d}\tau_1} = w'''(x) + 6w(x)w'(x) \tag{2.35b}$$

which coincide with the lowest two flows of the KdV hierarchy generated from the pseudodifferential operator, $L = \partial^2 + w$. In conclusion, we can regard equations of motion (2.29) and (2.30) as the lattice deformed KdV equations.

## 3. Lattice Boussinesq hierarchy

### *3.1. Lattice $W_3$ algebra*

We shall consider the deformed Boussinesq hierarchy, which should be formulated with the third-order difference operator ($N = 3$ in (1.6));

$$L = D^3 - t_1(z)D^2 + t_2(z)D - 1 = (D - \Lambda_1(z))(D - \Lambda_2(z))(D - \Lambda_3(z)) \tag{3.1}$$

where we set $\Lambda_1(z)\Lambda_2(z)\Lambda_3(z) = 1$, and the Poisson algebra is given as

$$\{\Lambda_i(z), \Lambda_i(w)\} = \sum_{m=-\infty}^{\infty} \left(\frac{w}{z}\right)^m \frac{(1-q^m)(1-q^{2m})}{1-q^{3m}} \Lambda_i(z)\Lambda_i(w) \qquad \text{for } i = 1, 2 \tag{3.2a}$$

$$\{\Lambda_1(z), \Lambda_2(w)\} = -\sum_{m=-\infty}^{\infty} \left(\frac{wq}{z}\right)^m \frac{(1-q^m)^2}{1-q^{3m}} \Lambda_1(z)\Lambda_2(w). \tag{3.2b}$$

The $q$-difference Miura transformations are written as

$$t_1(z) = \Lambda_1(z) + \Lambda_2(qz) + \Lambda_3(q^2z) \tag{3.3a}$$

$$t_2(z) = \Lambda_1(z)\Lambda_2(z) + \Lambda_1(z)\Lambda_3(qz) + \Lambda_2(qz)\Lambda_3(qz). \tag{3.3b}$$

Using the Poisson relations (3.2), these functions are shown to satisfy the following relations;

$$\{t_1(z), t_1(w)\} = \sum_{m\in\mathbb{Z}} \frac{(1-q^m)(1-q^{2m})}{1-q^{3m}} \left(\frac{w}{z}\right)^m t_1(z)t_1(w) + \delta\left(\frac{qw}{z}\right) t_2(z) - \delta\left(\frac{w}{qz}\right) t_2(w) \tag{3.4a}$$

$$\{t_1(z), t_2(w)\} = \sum_{m\in\mathbb{Z}} \frac{(1-q^m)^2}{1-q^{3m}} \left(\frac{w}{z}\right)^m t_1(z)t_2(w) + \delta\left(\frac{qw}{z}\right) - \delta\left(\frac{w}{q^2z}\right) \tag{3.4b}$$

$$\{t_2(z), t_2(w)\} = \sum_{m\in\mathbb{Z}} \frac{(1-q^m)(1-q^{2m})}{1-q^{3m}} \left(\frac{w}{z}\right)^m t_2(z)t_2(w) + \delta\left(\frac{qw}{z}\right) t_1(w) - \delta\left(\frac{w}{qz}\right) t_1(z). \tag{3.4c}$$

As a lattice analogue of these relations is known as the $q$-Boussinesq hierarchy, we introduce dynamical variables $u_k$ and $v_k$ on the lattice. These variables, $u_k$ and $v_k$, respectively correspond to the $q$-deformed free fields $\Lambda_1(zq^k)$ and $\Lambda_2(zq^k)$, and satisfy the Poisson relations,

$$\{u_k, u_l\} = \eta\varphi(k-l)u_k u_l \qquad \text{for } k > l \tag{3.5a}$$

$$\{v_k, v_l\} = \eta\varphi(k-l)v_k v_l \qquad \text{for } k > l \tag{3.5b}$$

$$\{u_k, v_l\} = \begin{cases} \eta\varphi(k-l+1)u_k v_l & \text{for } k > l \\ -\eta\varphi(k-l+1)u_k v_l & \text{for } k \leqslant l. \end{cases} \tag{3.5c}$$

Here a function $\varphi(k)$ denotes a *signature* of $k \in \mathbb{Z}$ modulo $N = 3$, and defined by

$$\varphi(k) = \varpi^k + \varpi^{2k} = \begin{cases} 2 & \text{for } k = 0 \pmod 3 \\ -1 & \text{otherwise} \end{cases} \tag{3.6}$$

with a primitive root of unity; $\varpi = \exp(2\pi\mathrm{i}/3)$. One sees the resemblance between the Poisson algebras, (3.2) and (3.5); the lattice algebra (3.5) is naively given by setting $q \to \varpi$

in (3.2). With these free variables $u_k$ and $v_k$ on the lattice, we consider a lattice analogue of the Miura transformation (3.3). We set dynamical variables $s_k$ and $t_k$ as

$$s_k = u_k + v_{k+1} + \frac{1}{u_{k+2}v_{k+2}} \tag{3.7a}$$

$$t_k = u_k v_k + \frac{1}{u_{k+1}} + \frac{u_k}{u_{k+1}v_{k+1}}. \tag{3.7b}$$

Using relations (3.5), we obtain the Poisson relations among $s_k$ and $t_k$ after tedious computation;

$$\{s_k, s_l\} = \eta\varphi(k-l)s_k s_l + 2\eta t_k \delta_{k,l+1} \qquad \text{for } k > l \tag{3.8a}$$

$$\{t_k, t_l\} = \eta\varphi(k-l)t_k t_l + 2\eta s_l \delta_{k,l+1} \qquad \text{for } k > l \tag{3.8b}$$

$$\{s_k, t_l\} = \begin{cases} -\eta\varphi(k-l-1)s_k t_l + 2\eta\delta_{k,l+1} & \text{for } k \geqslant l \\ \eta\varphi(k-l-1)s_k t_l - 2\eta\delta_{k,l-2} & \text{for } k < l. \end{cases} \tag{3.8c}$$

These relations are lattice analogues of the Poisson algebra (3.4) for the fields $t_1(z)$ and $t_2(z)$, and include 'long-range' connection. To erase the long-range connection in these Poisson relations, we introduce dynamical variables $L_k$ and $W_k$ by

$$W_k = \frac{1}{s_k s_{k+1} s_{k+2}} \tag{3.9a}$$

$$L_k = \frac{t_{k+1}}{s_k s_{k+1}}. \tag{3.9b}$$

After simple computation, we see that the non-trivial Poisson algebra among $L_k$ and $W_k$ is given as follows

$$\{W_{k+3}, W_k\} = 2\eta W_k W_{k+3} L_{k+2} \tag{3.10a}$$

$$\{W_{k+2}, W_k\} = 2\eta W_k W_{k+2}(-1 + L_{k+1} + L_{k+2}) \tag{3.10b}$$

$$\{W_{k+1}, W_k\} = 2\eta W_k W_{k+1}(-1 + L_k + L_{k+2}) \tag{3.10c}$$

$$\{L_{k+2}, L_k\} = 2\eta(L_k L_{k+1} L_{k+2} - L_k W_{k+1} - L_{k+2} W_k) \tag{3.10d}$$

$$\{L_{k+1}, L_k\} = 2\eta(L_k + L_{k+1} - 1)(L_k L_{k+1} - W_k) \tag{3.10e}$$

$$\{W_{k+2}, L_k\} = 2\eta W_{k+2}(-W_k + L_k L_{k+1}) \tag{3.10f}$$

$$\{W_{k+1}, L_k\} = 2\eta W_{k+1}(-W_k + L_k(L_k + L_{k+1} - 1)) \tag{3.10g}$$

$$\{W_k, L_k\} = 2\eta W_k(-W_k + L_k L_{k+1}) \tag{3.10h}$$

$$\{W_{k-1}, L_k\} = 2\eta W_{k-1}(W_{k-1} - L_{k-1} L_k) \tag{3.10i}$$

$$\{W_{k-2}, L_k\} = 2\eta W_{k-2}(W_{k-1} - L_k(L_k + L_{k-1} - 1)) \tag{3.10j}$$

$$\{W_{k-3}, L_k\} = 2\eta W_{k-3}(W_{k-1} - L_k L_{k-1}). \tag{3.10k}$$

This algebra was constructed in [12–14] as the lattice $W_3$ algebra. To see the relation with our construction, we introduce new variables $\alpha_k$ and $\beta_k$ as

$$\alpha_k = u_k u_{k+2} v_{k+2} \tag{3.11a}$$

$$\beta_k = u_{k+2} v_{k+1} v_{k+2}. \tag{3.11b}$$

These dynamical variables are shown to satisfy the following local Poisson relations from (3.5);

$$\{\alpha_{k+1}, \alpha_k\} = -2\eta\alpha_k\alpha_{k+1} \tag{3.12a}$$

$$\{\beta_{k+1}, \beta_k\} = -2\eta\beta_{k+1}\beta_k \tag{3.12b}$$

$$\{\alpha_k, \beta_k\} = -2\eta\alpha_k\beta_k \tag{3.12c}$$

$$\{\alpha_{k-1}, \beta_k\} = 2\eta\alpha_{k-1}\beta_k. \tag{3.12d}$$

From the definition of $L_k$ and $W_k$ (3.9), we find that variables $L_k$ and $W_k$ are written in terms of $\alpha_k$ and $\beta_k$ as

$$W_k = \frac{\beta_{k+1}\alpha_{k+2}}{(1+\alpha_k+\beta_k)(1+\alpha_{k+1}+\beta_{k+1})(1+\alpha_{k+2}+\beta_{k+2})} \tag{3.13a}$$

$$L_k = \frac{\alpha_{k+1}+\beta_{k+1}+\alpha_{k+1}\beta_k}{(1+\alpha_k+\beta_k)(1+\alpha_{k+1}+\beta_{k+1})}. \tag{3.13b}$$

It should be remarked that definition (3.13) of currents $L_k$ and $W_k$ are exactly same with the lattice Miura transformation in [12, 13], wherein the lattice $W_3$ algebra was studied in a different context.

### 3.2. Hamiltonian structure

As the LV model (2.29) was derived from the lattice Virasoro algebra, we shall consider the integrable models associated with the lattice $W_3$ algebra (3.10). For our purpose, we set the Lax matrix $\tilde{\mathbf{L}}_n(\lambda)$ as a $3\times 3$ matrix as

$$\tilde{\mathbf{L}}_n(\lambda) = \frac{1}{W_n^{1/3}}\begin{pmatrix} \lambda^2 & -\lambda L_{n+1} & W_n \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.14}$$

The determinant of the Lax matrix is set to be unity; $\mathrm{Det}\,\tilde{\mathbf{L}}_n(\lambda) = 1$. To prove the Poisson commutativity of the transfer matrix, we gauge transform the Lax matrix (3.14) as

$$\mathbf{L}_{n+1}(\lambda) = \boldsymbol{\Omega}_{n+1}(\lambda)\tilde{\mathbf{L}}_n(\lambda)\boldsymbol{\Omega}_n(\lambda)^{-1}. \tag{3.15}$$

Here matrix $\boldsymbol{\Omega}_n$ is given by

$$\boldsymbol{\Omega}_n = \begin{pmatrix} (\lambda^3-1)^{-1/3}\omega_n^{1/3}\omega_{n+1}^{2/3} & & \\ & \omega_n^{1/3}\omega_{n+1}^{-1/3} & \\ & & (\lambda^3-1)^{1/3}\omega_n^{-2/3}\omega_{n+1}^{-1/3} \end{pmatrix}$$

$$\times\begin{pmatrix} 1 & -\lambda^2(1+\beta_{n+1})\omega_{n+1}^{-1} & \lambda\beta_{n+1}\omega_n^{-1}\omega_{n+1}^{-1} \\ 0 & \lambda & -\omega_n^{-1} \\ 0 & \omega_n & 0 \end{pmatrix} \tag{3.16}$$

with $\omega_n \equiv 1+\alpha_n+\beta_n$. After a simple calculation, we obtain the local Lax matrix $\mathbf{L}_n(\lambda)$ as

$$\mathbf{L}_n(\lambda) = \begin{pmatrix} \lambda^2\alpha_{n+1}^{2/3}\beta_n^{-1/3} & (\lambda^3-1)^{2/3}\alpha_{n+1}^{2/3}\beta_n^{2/3} & \lambda(\lambda^3-1)^{1/3}\alpha_{n+1}^{2/3}\beta_n^{-1/3} \\ \lambda(\lambda^3-1)^{1/3}\alpha_{n+1}^{-1/3}\beta_n^{-1/3} & \lambda^2\alpha_{n+1}^{-1/3}\beta_n^{2/3} & (\lambda^3-1)^{2/3}\alpha_{n+1}^{-1/3}\beta_n^{-1/3} \\ (\lambda^3-1)^{2/3}\alpha_{n+1}^{-1/3}\beta_n^{-1/3} & \lambda(\lambda^3-1)^{1/3}\alpha_{n+1}^{-1/3}\beta_n^{2/3} & \lambda^2\alpha_{n+1}^{-1/3}\beta_n^{-1/3} \end{pmatrix}. \tag{3.17}$$

Here 'local' means that the non-trivial Poisson brackets are given as

$$\{\mathbf{L}_n(\lambda) \overset{\otimes}{,} \mathbf{L}_{n+1}(\mu)\} = 2\eta\mathbf{H}_1\mathbf{L}_n(\lambda)\otimes\mathbf{H}_1\mathbf{L}_{n+1}(\mu) + 2\eta(-\mathbf{H}_1\mathbf{L}_n(\lambda)+\mathbf{L}_n(\lambda)\mathbf{H}_2)\otimes\mathbf{L}_{n+1}(\mu)\mathbf{H}_2 \tag{3.18a}$$

$$\{\mathbf{L}_n(\lambda) \overset{\otimes}{,} \mathbf{L}_{n+2}(\mu)\} = 2\eta\mathbf{H}_1\mathbf{L}_n(\lambda)\otimes\mathbf{L}_{n+2}(\mu)\mathbf{H}_2 \tag{3.18b}$$

where matrices $\mathbf{H}_1$ and $\mathbf{H}_2$ are diagonal,

$$\mathbf{H}_1 = \begin{pmatrix} -\frac{2}{3} & & \\ & \frac{1}{3} & \\ & & \frac{1}{3} \end{pmatrix} \qquad \mathbf{H}_2 = \begin{pmatrix} \frac{1}{3} & & \\ & -\frac{2}{3} & \\ & & \frac{1}{3} \end{pmatrix}.$$

As in the case of the FTV algebra, we define the monodromy matrix $\mathbf{T}(\lambda)$ and the transfer matrix $t(\lambda)$ as

$$\mathbf{T}(\lambda) = \prod_n^{\curvearrowleft} \mathbf{L}_n(\lambda) \tag{3.19}$$

$$t(\lambda) = \mathrm{Tr}\, \mathbf{T}(\lambda). \tag{3.20}$$

We see from the Poisson algebra (3.18) that, if we have the classical $r$-matrix satisfying,

$$\begin{aligned}
&\mathbf{L}_n(\lambda)\mathbf{H}_2 \otimes \mathbf{L}_n(\mu)\mathbf{H}_1 + \mathbf{L}_n(\lambda)\mathbf{H}_2 \otimes \mathbf{H}_2\mathbf{L}_n(\mu) + \mathbf{L}_n(\lambda)\mathbf{H}_1 \otimes \mathbf{H}_2\mathbf{L}_n(\mu) + \mathbf{L}_n(\lambda)\mathbf{H}_1 \otimes \mathbf{H}_1\mathbf{L}_n(\mu) \\
&\quad -\mathbf{H}_2\mathbf{L}_n(\lambda) \otimes \mathbf{L}_n(\mu)\mathbf{H}_2 - \mathbf{H}_2\mathbf{L}_n(\lambda) \otimes \mathbf{L}_n(\mu)\mathbf{H}_1 \\
&\quad -\mathbf{H}_1\mathbf{L}_n(\lambda) \otimes \mathbf{L}_n(\mu)\mathbf{H}_1 - \mathbf{L}_n(\lambda)\mathbf{H}_1 \otimes \mathbf{L}_n(\mu)\mathbf{H}_2 \\
&\quad = \tfrac{1}{2}[\mathbf{r}(\lambda,\mu), \mathbf{L}_n(\lambda) \otimes \mathbf{L}_n(\mu)]
\end{aligned} \tag{3.21}$$

we have a usual Poisson structure for the monodromy matrix $\mathbf{T}(\lambda)$ as,

$$\{\mathbf{T}(\lambda) \overset{\otimes}{,} \mathbf{T}(\mu)\} = \eta[\mathbf{r}(\lambda,\mu), \mathbf{T}(\lambda) \otimes \mathbf{T}(\mu)]. \tag{3.22}$$

This Poisson structure for the monodromy matrix proves that the transfer matrix $t(\lambda)$ becomes a generating function of the conserved quantities for an integrable model associated with the lattice $W_3$ algebra;

$$\{t(\lambda), t(\mu)\} = 0. \tag{3.23}$$

Indeed we empirically find that equation (3.21) is fulfilled for the following classical $r$-matrix;

$$\mathbf{r}(\lambda,\mu) = \left(\begin{array}{ccc|ccc|ccc}
0 & & & & & & & & \\
 & a & & c & & & & & \\
 & & b & & & & d & & \\
\hline
 & d & & b & & & & & \\
 & & & & 0 & & & & \\
 & & & & & a & & c & \\
\hline
 & & c & & & & a & & \\
 & & & & & d & & b & \\
 & & & & & & & & 0
\end{array}\right) \tag{3.24}$$

where each matrix element is given by

$$a = \frac{1}{3} - \frac{\theta + \theta^{-1}}{\theta - \theta^{-1}} \qquad b = -\frac{1}{3} - \frac{\theta + \theta^{-1}}{\theta - \theta^{-1}}$$

$$c = \frac{2\theta^{-1/3}}{\theta - \theta^{-1}} \qquad d = \frac{2\theta^{1/3}}{\theta - \theta^{-1}}$$

with a modified spectral parameter $\theta$,

$$\theta = \sqrt{\frac{1-\mu^{-3}}{1-\lambda^{-3}}}.$$

We remark that this $r$-matrix is $\mathbb{Z}_3$-invariant,

$$\begin{aligned}
&\mathbf{r}(\lambda,\mu) = (\mathbf{C} \otimes \mathbf{C})\mathbf{r}(\lambda,\mu)(\mathbf{C}^{-1} \otimes \mathbf{C}^{-1}) \\
&\mathbf{C} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{C}^3 = \mathbb{1}.
\end{aligned} \tag{3.25}$$

One sees that the $r$-matrix (3.24) satisfies the classical Yang–Baxter equation, and that it is a classical analogue of the $\mathbb{Z}_3$-invariant solution of the Yang–Baxter equation [17]. This fact reminds us of a construction of the $W_N$ algebra from the $\mathbb{Z}_N$ invariant conformal field theory [18].

We extract the integrable Hamiltonians for the lattice $W_3$ algebra (3.10). The conserved quantities are given from the transfer matrix by

$$t(\lambda) = \exp(-\mathcal{H}_0)(\lambda^{2M} + \mathcal{H}_1\lambda^{2M-3} + \mathcal{H}_2'\lambda^{2M-6} + \mathcal{H}_3'\lambda^{2M-9} + \ldots) \tag{3.26}$$

where, as before, $M$ is the number of lattices. Due to the gauge transformation (3.15), each conserved quantities can be given in terms of the $W$-algebra currents $W_n$ and $L_n$. We note that each Hamiltonian is computed as follows

$$\mathcal{H}_0 = \tfrac{1}{3}\sum_n \log W_n \tag{3.27a}$$

$$\mathcal{H}_1 = -\sum_n L_n \tag{3.27b}$$

$$\begin{aligned}\mathcal{H}_2 &= \tfrac{1}{2}(\mathcal{H}_1)^2 - \mathcal{H}_2' \\ &= \sum_n (\tfrac{1}{2}L_n^2 + L_nL_{n+1} - W_n)\end{aligned} \tag{3.27c}$$

$$\begin{aligned}\mathcal{H}_3 &= \mathcal{H}_1\mathcal{H}_2' - \tfrac{1}{3}(\mathcal{H}_1)^3 - \mathcal{H}_3' \\ &= \sum_n (\tfrac{1}{3}L_n^3 + L_nL_{n+1}(L_n + L_{n+1} + L_{n+2}) - W_n(L_{n-1} + L_n + L_{n+1} + L_{n+2})).\end{aligned} \tag{3.27d}$$

By construction, these Hamiltonians Poisson commute with each other. As time evolution flows associated with these Hamiltonians, we define the time $t_m$ as

$$\frac{\mathrm{d}\mathcal{O}}{\mathrm{d}t_m} = \{\mathcal{H}_m, \mathcal{O}\}. \tag{3.28}$$

We then obtain the equations of motions for $t_0$ as

$$\frac{\mathrm{d}L_n}{\mathrm{d}t_0} = 2\eta(-W_n + W_{n-1} + L_n(L_{n+1} - L_{n-1})) \tag{3.29a}$$

$$\frac{\mathrm{d}W_n}{\mathrm{d}t_0} = 2\eta W_n(L_{n+2} - L_{n-1}). \tag{3.29b}$$

Equations of motions for the Hamiltonian $\mathcal{H}_1$ are computed as,

$$\begin{aligned}\frac{\mathrm{d}L_n}{\mathrm{d}t_1} &= 2\eta((W_n - L_nL_{n+1})(L_n + L_{n+1} + L_{n+2} - 1) \\ &\quad -(W_{n-1} - L_nL_{n-1})(L_n + L_{n-1} + L_{n-2} - 1) + L_n(W_{n+1} - W_{n-2}))\end{aligned} \tag{3.30a}$$

$$\begin{aligned}\frac{\mathrm{d}W_n}{\mathrm{d}t_1} &= 2\eta W_n(W_{n+1} + W_{n+2} - W_{n-1} - W_{n-2} - L_{n+2}(L_{n+1} + L_{n+2} + L_{n+3} - 1) \\ &\quad +L_{n-1}(L_n + L_{n-1} + L_{n-2} - 1)).\end{aligned} \tag{3.30b}$$

As will be clarified in section 3.3, these differential–difference equations can be regarded as the lattice Boussinesq equations.

### 3.3. Continuum limit

We shall take the continuum limit of the dynamical equations on the lattice, (3.29) and (3.30). As in the case of the lattice KdV hierarchy, we set the free dynamical variables $u_n$

and $v_n$ (3.5) as

$$u_{n+i} \to \exp(\epsilon r_1(x - \mathrm{i}\epsilon)) \tag{3.31$a$}$$

$$v_{n+i} \to \exp(\epsilon r_2(x - \mathrm{i}\epsilon)) \tag{3.31$b$}$$

where $r_1(x)$ and $r_2(x)$ are the free fields in (1.1). In this limit we find that, using the definitions (3.9) and (3.7), the variables $L_n$ and $W_n$ reduce to

$$L_n \to \frac{1}{3} + \frac{\epsilon^2}{9}\left(w_1(x) + \frac{\epsilon}{2}\tilde{w}_1(x)\right) + \mathrm{O}(\epsilon^4) \tag{3.32$a$}$$

$$W_n \to \frac{1}{27} + \frac{\epsilon^2}{27}\left(w_1(x) + \frac{\epsilon}{2}\tilde{w}_1(x)\right) + \frac{\epsilon^3}{27}(w_2(x) - w_1'(x)) + \mathrm{O}(\epsilon^4). \tag{3.32$b$}$$

Here $w_1(x)$ and $w_2(x)$ denote fields in the pseudodifferential operator $L$ (1.2), and explicitly written as,

$$w_1(x) = -r_1^2 - r_1 r_2 - r_2^2 - 2r_1' - r_2' \tag{3.33$a$}$$

$$w_2(x) = -r_1^2 r_2 - r_1 r_2^2 - (r_1 + r_2)(r_1' + 2r_2') - r_1'' - r_2''. \tag{3.33$b$}$$

These are the generalized Miura transformations. Field $\tilde{w}_1(x)$ is an ‘unwanted’ field given by

$$\tilde{w}(x) = 3(r_1 r_2 + 2r_1')(r_1 + r_2) + (8r_1 + 10r_2)r_2' + 6r_1'' + 5r_2''. \tag{3.34}$$

To write down the continuum limit of the integrable lattice equations (3.29) and (3.30), we re-scale the time flows $t_m$ (3.28) as

$$\frac{\mathrm{d}}{\mathrm{d}t_0} \to -2\eta\frac{\epsilon}{3}\frac{\mathrm{d}}{\mathrm{d}\tau_0} \tag{3.35$a$}$$

$$\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t_0} + \frac{\mathrm{d}}{\mathrm{d}t_1} \to 2\eta\frac{\epsilon^2}{9}\frac{\mathrm{d}}{\mathrm{d}\tau_1}. \tag{3.35$b$}$$

After a straightforward but lengthy computation, we obtain the continuum limit of the lattice integrable models (3.29) and (3.30) as follows

$$\frac{\mathrm{d}w_1(x)}{\mathrm{d}\tau_0} = w_1'(x) \tag{3.36$a$}$$

$$\frac{\mathrm{d}w_2(x)}{\mathrm{d}\tau_0} = w_2'(x) \tag{3.36$b$}$$

$$\frac{\mathrm{d}w_1(x)}{\mathrm{d}\tau_1} = w_1'' - 2w_2' \tag{3.37$a$}$$

$$\frac{\mathrm{d}w_2(x)}{\mathrm{d}\tau_1} = -w_2'' + \frac{2}{3}w_1''' + \frac{2}{3}w_1 w_1'. \tag{3.37$b$}$$

These are nothing but the first two flows of the Boussinesq hierarchy generated from the third-order pseudodifferential operator, $L = \partial^3 + w_1\partial + w_2$.

## 4. Concluding remarks

In this paper we have studied the lattice analogue of the $W_N$ algebra. Following a method of [15], we have introduced integrable models in terms of the free dynamical variables on the lattice. It is shown that, in the case of the $N = 2$ case, the integrable model is the famous LV model. As a generalization, we have studied the lattice $W_3$ algebra in detail. We have checked that our construction from the free dynamical variables produces Belov *et* *al*'s lattice algebra. The integrable structure of the general $W_N$ algebra would be discussed in the forth coming paper [19].

We have only treated the *classical* algebra in this paper, but the quantization of the *local* $2 \times 2$ Lax matrix for the lattice KdV hierarchy (2.20) was studied in [10, 20–22], and enlightens the structure of the sine-Gordon model on the lattice and the discrete KdV equation. In this sense, the quantization of the *local* Lax matrix for the lattice Boussinesq hierarchy (3.17) and the $N$-reduced lattice KP hierarchy will give us new insights into the discretization of the conformal field theories [23].

As a final comment, there exists the Bogoyavlensky model (or, the hungry Volterra model) [24–27], which was introduced as a discretization of the KdV equation;

$$\frac{\mathrm{d}V_n}{\mathrm{d}\tau} = V_n\left(\sum_{i=1}^{M} V_{n+i} - \sum_{i=1}^{M} V_{n-i}\right). \tag{4.1}$$

In the simple case $M = 1$, this model reduces to the LV model (2.29), and currents $V_n$ constitute the FTV algebra, i.e. the lattice Virasoro algebra. Further, equations of motion (3.30) can be reproduced from those for the Bogoyavlensky model with $M = 2$. Based on these facts, we may regard the Bogoyavlensky model (4.1) as a simple realization of the lattice $W_{M+1}$ algebra.

*Note added in proof.* Using the $\tau$-function we find that the Bogoyavlensky lattice (4.1) is equivalent to the lattice $W_{M+1}$ algebra.

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